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Projection Theorems for Intermediate Dimensions

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Motivation

In 1954 Marstrand proved that the Hausdorff dimension of projections of Borel $E \subseteq \mathbb{R}^2$ satisfy

$$\dim_{\mathrm{H}} \mathrm{proj}_V E = \min\{\dim_{\mathrm{H}} E, 1\},$$

for almost all one-dimensional subspaces $V \subseteq \mathbb{R}^2$, where proj_V denotes orthogonal projection onto V. Later on, this was extended to higher dimensions by Matilla. This motivated us to ask whether an analogous theorem were true for the intermediate dimensions introduced by Falconer, Fraser and Kempton in 2018.

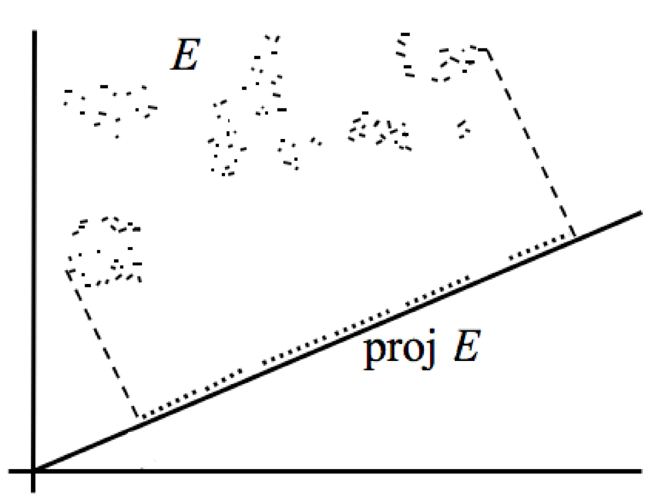

Figure 1: An example projection of a set in $\mathbb{R}^2$ onto a linear subspace.

Intermediate Dimensions

Intermediate dimensions refer to a spectrum of dimensions that interpolate between the Hausdorff and box-counting dimensions of a set E. To gain an intuitive appreciation for their definition, first recall that:

- Box dimension is derived from the growth rate of the cardinality of covers of sets of equal diameter as this diameter shrinks to zero.
- Hausdorff dimension is defined by

$$\dim_{\mathrm{H}} E = \inf\Big\{s \geq 0 : \ \forall \epsilon > 0 \colon \exists \text{ a cover } \{U_i\}_i \text{ of } F \text{ such that } \textstyle\sum |U_i|^s \leq \epsilon\Big\},$$

where no restriction on the relationships of diameters within a cover is given.

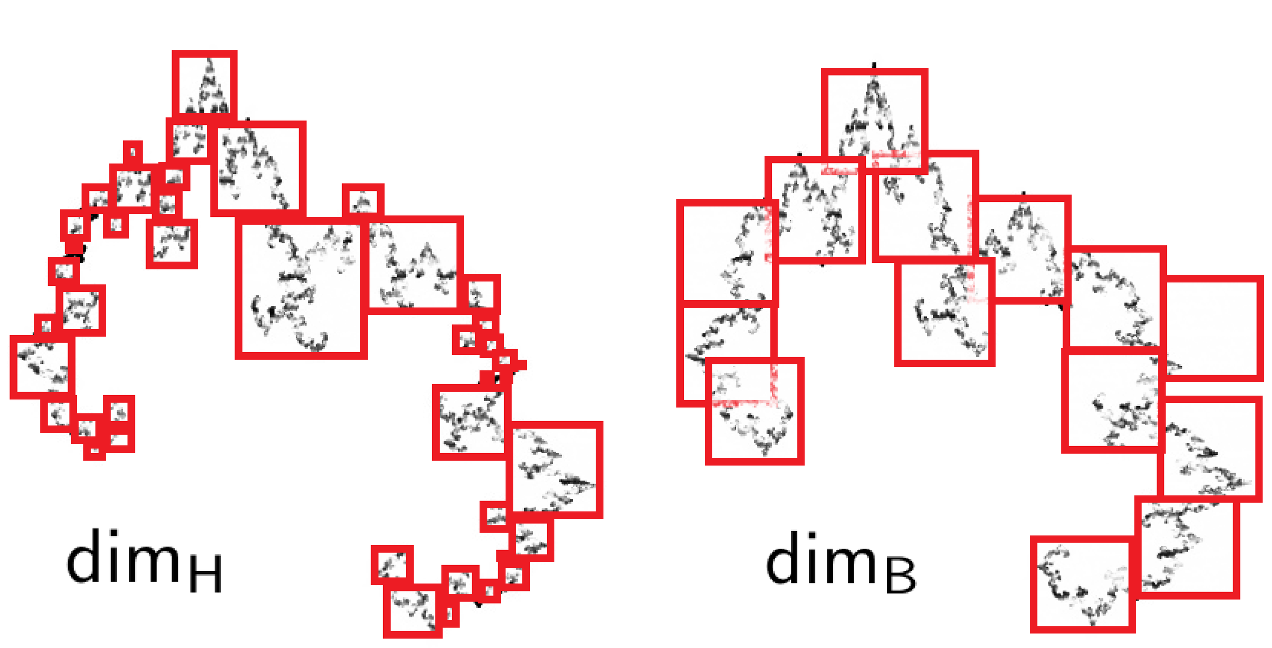

Figure 2: The types of covers considered by Box and Hausdorff dimensions.

This motivates the intermediate dimensions: for $\theta \in [0,1]$, the θ-intermediate dimension $\dim_\theta F$ is defined in essentially the same way as $\dim_{\mathrm{H}} E$, except each pair of sets U, V within the covers must satisfy $|U| \leq |V|^\theta$. Notice how this recovers the box-counting and Hausdorff covering schemes when $\theta = 1$ and $\theta = 0$, respectively.

Capacities and Kernels

Let $E \subset \mathbb{R}^n$ be Borel, $\theta \in (0,1]$, $m \in \{1, \dots, n\}$, $0 \leq s \leq m$ and $0 < r < 1$. For a potential kernel $\phi_{r,\theta}^{s,m} : \mathbb{R}^n \to \mathbb{R}^+$, the *capacity* $C_{r,\theta}^{s,m}(E)$ of E is

$$C_{r,\theta}^{s,m}(E) = \left(\inf_{\mu \in \mathcal{M}(E)} \int\int \phi_{r,\theta}^{s,m}(x-y)\, d\mu(x) d\mu(y) \right)^{-1},$$

i.e. the reciprocal of the minimum energy achieved by Borel measures supported on E.

The choice of kernel is crucial. For intermediate dimensions, we discovered

$$\phi_{r,\theta}^{s,m}(x) = \begin{cases} 1 & 0 \leq |x| < r \\ \big(r/|x|\big)^s & r \leq |x| < r^\theta \\ r^{\theta(m-s)+s}/|x|^m & r^\theta \leq |x| \end{cases} \qquad (x \in \mathbb{R}^n)$$

was effective.

Dimension Profiles

For a general notion of dimension and $m \in \{1, \dots, n\}$, an m-dimension profile aims to provide a set function whose image of a set E is the almost-sure value of the dimension of orthogonal projections of E onto m-dimensional subspaces. For example, box dimension profiles were introduced by Falconer and Howroyd in 1996.

In our setting, we define the *m-intermediate dimension profile* of $E \subset \mathbb{R}^n$, in terms of capacities, as

$$\dim_\theta^m E = \text{ the unique } s \in [0,m] \text{ such that } \lim_{r \to 0} \frac{\log C_{r,\theta}^{s,m}(E)}{-\log r} = s.$$

Results

Our first main result shows that the dimension profiles recover the intermediate dimensions when $m = n$.

Theorem 1 If $E \subset \mathbb{R}^n$ is compact and $\theta \in (0,1]$, then

$$\dim_\theta E = \dim_\theta^n E.$$

Our second result is a Marstrand-type theorem for the intermediate dimensions. In the following, $G(n,m)$ denotes the Grassmanian of m-dimensional linear subspaces of $\mathbb{R}^n$ and $\gamma_{n,m}$ the natural invariant probability measure on $G(n,m)$.

Theorem 2 If $E \subset \mathbb{R}^n$ is bounded, then

$$\dim_\theta \mathrm{proj}_V E \leq \dim_\theta^m E$$

for all $\theta \in (0,1]$ and all $V \in G(n,m)$. Moreover, for $\gamma_{n,m}$-almost all $V \in G(n,m)$,

$$\dim_\theta \mathrm{proj}_V E = \dim_\theta^m E$$

for all $\theta \in (0,1]$.

An intuitive way of understanding Theorem 2 is that our dimension profiles provide the almost-sure intermediate dimension of E when seen from an m-dimensional viewpoint.